

Spurious causality hiding an action at a distance

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Abstract

The Coulomb/Gauss law given in the Maxwell equations describes a spatial relation between the electric field component $E_{\parallel}(\mathbf{r}, t)$ and its source $\rho(\mathbf{r}, t)$ that is instantaneous, occurring at the same time t . This instantaneous action at a distance can be hidden by writing it formally as the sum of two causal terms. I show here that the causal expression of the total electric field \mathbf{E} suffers from such a spurious causality hiding the action at a distance that is E_{\parallel} . In fact, E_{\parallel} remains instantaneous in time even when it is part of a wave equation. Only the \mathbf{E}_{\perp} part is really causal.

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I. THE PUZZLE

It is well known that far from charges and currents, the electromagnetic (EM) fields \mathbf{E} and \mathbf{B} satisfy homogeneous vacuum wave equations where the speed of light c appears. These wave equations involve only the transverse parts of the EM fields.

The situation is more complicated when the system is close to charges and currents. The vacuum wave equations are then inhomogeneous. Most textbooks derive directly from the Maxwell equations the inhomogeneous wave equations for these EM fields, using the Lorenz gauge first introduced by L. V. Lorenz in 1867.¹ Many textbooks also express the scalar and vector potentials separately as retarded solutions relative to their sources, namely the charge and current densities, respectively. These expressions were also first used by Lorenz in 1867.¹⁻³

More recently, Panofsky and Phillips⁴ and Jefimenko^{5,6} have given expressions for the entire \mathbf{E} and \mathbf{B} fields where every term is retarded with respect to the charge-current sources. The results are considered “time-dependent generalizations of the Coulomb and Biot-Savart laws.”^{2,7,8}

Yet the electric field \mathbf{E} has an identifiable part that satisfies the original vacuum Coulomb/Gauss law in the presence of a time-dependent source

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \rho(\mathbf{r}, t)/\epsilon_0 \quad (1)$$

in SI units. This Maxwell equation states unambiguously that the part of the field \mathbf{E} involved in the equation is nonlocally related to its source ρ in space, but it occurs at the same time or instantaneously as the source ρ .^{9,10}

That the result involves only a part of \mathbf{E} can be made manifest by working in the Fourier space \mathbf{k} instead of the real space \mathbf{r} . The notation and results of Wong⁹ will be used here to avoid duplication. In the Fourier space, the Coulomb/Gauss law reads

$$\tilde{E}_{\parallel}(\mathbf{k}, t) = -\frac{i}{\epsilon_0 k} \tilde{\rho}(\mathbf{k}, t). \quad (2)$$

Thus only \tilde{E}_{\parallel} is instantaneous in time.

As shown in Ref. 9, Eq. (2) appears twice in the Maxwell equations. It can also be obtained a second time as the longitudinal part of the Maxwell $\nabla \times \mathbf{B}$ equation

$$\tilde{\mathbf{J}}_{\parallel}(\mathbf{k}, t) + \epsilon_0 \partial_t \tilde{\mathbf{E}}_{\parallel}(\mathbf{k}, t) = 0, \quad (3)$$

when use is made of the continuity equation. Here $\partial_t = \partial/\partial t$. Equation (3) thus provides a counterexample to the claim of Ref. 7 that the two sources \mathbf{J} and $\partial_t \mathbf{E}$ of the $\nabla \times \mathbf{B}$ equation are completely independent of each other.

Can the electric field \mathbf{E} be causal as a whole, and yet instantaneous in its longitudinal part? It is puzzling how an identifiable part \mathbf{E}_{\parallel} of the electric field that is an action at a distance can disappear from plain sight, like the tiger in a magician show. The purpose of this paper is to show that the trick is done by writing it as a sum of two causal terms, and that \mathbf{E}_{\parallel} itself, though cut up and disguised, remains instantaneous in time. In Sect. II, I shall do this by working directly with electric fields without using scalar and vector potentials in order to preserve manifest gauge independence.

The same puzzle does not appear in \mathbf{B} . With no monopole present, $\mathbf{B}_{\parallel} = 0$. Therefore $\mathbf{B} = \mathbf{B}_{\perp}$ is entirely transverse and causal. It is of course well defined both mathematically and physically. \mathbf{E}_{\perp} is intimately related to \mathbf{B}_{\perp} . They drive each other forward as an electromagnetic wave through Faraday induction and Maxwell displacement. So \mathbf{E}_{\perp} too is well defined both mathematically and physically. Finally $\mathbf{E}_{\parallel} = \mathbf{E} - \mathbf{E}_{\perp}$ is well defined mathematically and physically.

There have been at least two previous attempts to explain why the entire \mathbf{E} might be causal.^{11,12} Both attempts make use of the causal form of the scalar and vector potentials in the Lorenz gauge. We shall show in Sect. III that both ρ and A_{\parallel} suffer from spurious causality, the same disease that infects the causal form of E_{\parallel} .

Finally, the Maxwell equations and their Fourier representation are given in appendix A for the reader's convenience.

II. SPURIOUS CAUSALITY IN THE ELECTRIC FIELD

Spurious causality can be introduced into the instantaneous action at a distance that is $\mathbf{E}_{\parallel}(\mathbf{r}, t)$ by working in the full Fourier space (\mathbf{k}, ω) . Equation (2) can then be written as

$$\tilde{E}_{\parallel}(\mathbf{k}, \omega) = -\frac{i}{\epsilon_0} \frac{\tilde{\rho}}{k} \left(\frac{k^2 - \omega^2/c^2}{k^2 - \omega^2/c^2} \right). \quad (4)$$

Note that the two introduced factors depend on c and are therefore causal. They obviously cancel each other so that the entire expression remains instantaneous in time on going back to the time description.

It is possible to hide this cancellation by using the continuity equation for a conserved charge density

$$\begin{aligned} 0 &= \frac{d\rho}{dt} = \partial_t \rho + \mathbf{v} \cdot \nabla \rho \\ &= \partial_t \rho + \nabla \cdot \mathbf{J}, \end{aligned} \quad (5)$$

where $\mathbf{v} = d\mathbf{r}/dt$ and $\mathbf{J} \equiv \mathbf{v}\rho$. In the Fourier space (\mathbf{k}, ω) , the continuity equation takes the more transparent form

$$\omega \tilde{\rho} = k \tilde{J}_{\parallel}. \quad (6)$$

It shows that only the longitudinal component J_{\parallel} in space-time is involved in charge conservation. The transverse component remains irreducible in its original $\mathbf{v}\rho$ form.

Using the continuity equation, one finds in Eq. (4) a factor

$$\frac{\tilde{\rho}}{k}(k^2 - \omega^2/c^2) = \tilde{\rho}k - \tilde{J}_{\parallel}\omega\epsilon_0\mu_0. \quad (7)$$

Hence

$$\begin{aligned} \tilde{E}_{\parallel}(\mathbf{k}, \omega) &= -\frac{i}{\epsilon_0(k^2 - \omega^2/c^2)}(\tilde{\rho}k - \tilde{J}_{\parallel}\omega\epsilon_0\mu_0) \\ &= \tilde{E}_{J_{\parallel}\text{causal}} + \tilde{E}_{J_{\parallel}}. \end{aligned} \quad (8)$$

The final two terms are given the notation used in Ref. 10. Each term is causal, propagating with light speed c .

The causality so introduced is totally spurious, however, because their sum is already known to be instantaneous. That is, the introduced causality effects must cancel out *completely* between the two terms. In fact, we could have used a nonphysical and arbitrary gauge velocity v_g here, with exactly the same cancellation, as will be done in section III. This is just another way of showing that the causality effects involved are really spurious. Doing it with the physical light speed c , however, seems to have endowed these spurious terms with real physical properties.

Spurious causality also infects the vacuum wave equation for the electric field \mathbf{E} in the presence of sources

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\mathbf{E} = \frac{1}{\epsilon_0}\nabla\rho + \mu_0\partial_t\mathbf{J}. \quad (9)$$

Decomposed into longitudinal and transverse parts, conveniently done in the Fourier space (\mathbf{k}, ω) , it gives

$$\left(k^2 - \frac{\omega^2}{c^2}\right) \tilde{\mathbf{E}}_{\perp} = i\mu_0\omega\tilde{J}_{\perp}, \quad (10)$$

$$\begin{aligned} \left(k^2 - \frac{\omega^2}{c^2}\right) \tilde{\mathbf{E}}_{\parallel} &= -i\frac{k\tilde{\rho}}{\epsilon_0} + i\mu_0\omega\tilde{J}_{\parallel} \\ &= -\frac{i}{\epsilon_0}\frac{\tilde{\rho}}{k}\left(k^2 - \frac{\omega^2}{c^2}\right). \end{aligned} \quad (11)$$

Use has been made of the identity (7) to reach the last expression. The two factors of $(k^2 - \omega^2/c^2)$ cancel in the wave equation for $\tilde{\mathbf{E}}_{\parallel}$, leaving the instantaneous result of Eq. (2). The myth that the entire \mathbf{E} is causal, a myth passed down by generations of textbooks since 1867, can finally be laid to rest.

No rearrangements can change these properties. For example, in the wave equation (10), one can separate the source current into parts $\tilde{J}_{\perp} = \tilde{J} - \tilde{J}_{\parallel}$ to give the separation $\tilde{\mathbf{E}}_{\perp} = \tilde{\mathbf{E}}_J - \tilde{\mathbf{E}}_{J\parallel}$, as done in Ref. 10. Then

$$\mathbf{E} = \mathbf{E}_{\perp} + \mathbf{E}_{\parallel} = (\mathbf{E}_J - \mathbf{E}_{J\parallel}) + (\mathbf{E}_{J\parallel} + \mathbf{E}_{J\parallel\text{causal}}). \quad (12)$$

The final sum $\mathbf{E} = \mathbf{E}_J + \mathbf{E}_{J\parallel\text{causal}}$ of two nominally causal terms actually contains the instantaneous \mathbf{E}_{\parallel} term, but this instantaneous component has been cut up and disguised as causal. Since the longitudinal/transverse separation is unique, the resulting causal/instantaneous separation is also unique and beyond doubt.

To summarize, I have verified the result of Ref. 9 that the total electric field is made up of a unique causal transverse part \mathbf{E}_{\perp} , containing physical signals traveling at the physical light speed, and a unique longitudinal part \mathbf{E}_{\parallel} that is an instantaneous action at a distance.

The method of Ref. 9 is cleaner than the demonstration given here because the longitudinal part of Maxwell's displacement current is separated from the Maxwell $\nabla \times \mathbf{B}$ equation before the rest of the equation is combined with the $\nabla \times \mathbf{E}$ equation to give the transverse wave equations. In this way, the spurious causality seen in Eq. (11) is entirely avoided. The separated longitudinal displacement current can then be combined with the continuity equation to give the instantaneous Coulomb/Gauss law a second time, as already discussed in connection with Eq. (3).

III. SPURIOUS CAUSALITY IN SCALAR AND VECTOR POTENTIALS

In the Fourier space (\mathbf{k}, ω) , the transverse fields can be described uniquely by a transverse vector potential:

$$\tilde{\mathbf{B}}_{\perp} = i\mathbf{k} \times \tilde{\mathbf{A}}_{\perp}, \quad \text{where} \quad \tilde{\mathbf{A}}_{\perp} = \frac{i}{k^2} \mathbf{k} \times \tilde{\mathbf{B}}_{\perp} \quad (13)$$

$$\tilde{\mathbf{E}}_{\perp} = i\omega \tilde{\mathbf{A}}_{\perp}. \quad (14)$$

Like $\tilde{\mathbf{B}}_{\perp}$ and $\tilde{\mathbf{E}}_{\perp}$, $\tilde{\mathbf{A}}_{\perp}$ is causal, propagating with light speed c . In classical electrodynamics, one can stop at this point and never run into any redundant gauge degree of freedom.

However, the idea of a potential, so successful in classical mechanics, was too good to pass up. So it developed that the instantaneous longitudinal electric field was given an alternative gauge-dependent form

$$\begin{aligned} \tilde{E}_{\parallel} &= -\frac{i}{\epsilon_0} \frac{\tilde{\rho}}{k} \\ &= -ik\tilde{\Phi} + i\omega\tilde{A}_{\parallel}, \end{aligned} \quad (15)$$

where a single field component is described redundantly by two scalar fields $\tilde{\Phi}$ and \tilde{A}_{\parallel} . The redundancy has to be removed by a choice of gauge. For example, velocity gauges are defined by the gauge condition^{9,13,14}

$$\tilde{A}_{\parallel}^{(vg)} = \alpha \frac{\omega}{k} \tilde{\Phi}^{(vg)}, \quad (16)$$

where the gauge parameter $\alpha = 1/v_g^2$ can be written in term of a gauge velocity v_g . With two algebraic equations (15) and (16) in two unknowns $\tilde{\Phi}$, \tilde{A}_{\parallel} , one can solve for them to get the causal expressions

$$\begin{aligned} \tilde{\Phi}^{(vg)} &= \frac{\tilde{\rho}}{\tilde{\epsilon}_{\parallel}(k^2 - \omega^2/v_g^2)}, \\ \tilde{A}_{\parallel}^{(vg)} &= \frac{\omega}{kv_g^2} \frac{\tilde{\rho}}{\tilde{\epsilon}_{\parallel}(k^2 - \omega^2/v_g^2)} \\ &= \frac{1}{v_g^2} \frac{\tilde{J}_{\parallel}}{\tilde{\epsilon}_{\parallel}(k^2 - \omega^2/v_g^2)}. \end{aligned} \quad (17)$$

In this way, an action at a distance \tilde{E}_{\parallel} can be cut up and disguised as two causal terms.

The disguise is most effective when $v_g = c$ is used. Combined with the really causal transverse vector potential $\tilde{\mathbf{A}}_{\perp}$, one finds Lorenz's causal expressions for $\tilde{\rho}$ and $\tilde{\mathbf{A}}$ given in

many textbooks. These potentials are usually obtained by solving the inhomogeneous wave equations in the Lorenz gauge for these potentials.

The causality they display is totally spurious, however. When substituted into Eq. (15), these causal potential components simply give

$$\tilde{E}_{\parallel} = \frac{k^2}{k^2 - \omega^2/v_g^2} \tilde{E}_{\parallel} - \frac{\omega^2/v_g^2}{k^2 - \omega^2/v_g^2} \tilde{E}_{\parallel}, \quad (18)$$

The result shows clearly that the introduced spurious causality is canceled out completely by the numerator terms. It matter not a whit if $v_g = c$, as used in the Lorenz gauge, or if $v_g \neq c$ and therefore obviously unphysical. Only when $v_g \rightarrow \infty$, do we recover the original instantaneous, non-causal result, free of any masquerading causal terms.

The original result is just the result obtained in the Coulomb gauge. It is also the gauge-independent result that comes directly from the Coulomb/Gauss part of the Maxwell equations. So this instantaneous result cannot be avoided. In fact, Eq. (18) has been used in Ref. 9 to verify the gauge invariance of \tilde{E}_{\parallel} when constructed from the $\tilde{\rho}$ and \tilde{A}_{\parallel} of any velocity gauge, including the Lorenz gauge.

This is not to say that Lorenz-gauge expressions for the potentials and \mathbf{E} are not practically useful and convenient. By using the same causality everywhere, they achieve great compactness and ease of handling. Well-known examples include the Lienard-Wiechert potential of 1898-1900¹⁵ and Jefimenko's expression for \mathbf{E} . The only caveat is that when it comes to interpretation, the instantaneous nature of \mathbf{E}_{\parallel} should be acknowledged if not made explicit.

In conclusion, we have verified the result of Ref. 9 that the so-called time-dependent generalizations of the Coulomb and Biot-Savart laws and the causal interpretation of ρ and \mathbf{A}_{\parallel} are mathematically misleading and physically meaningless.

Appendix A: Maxwell equations and its Fourier representation

For the reader's convenience, we reproduce here the vacuum Maxwell equations for electromagnetic fields in space-time (\mathbf{r}, t) in SI units in the notation of Jackson (p. 248):⁸

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \quad \nabla \cdot \mathbf{B} = 0, \quad (A1)$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \partial_t \mathbf{E}/c^2. \quad (A2)$$

Here $\partial_t \equiv \partial/\partial t$, while $\rho = \rho(\mathbf{r}, t)$ and \mathbf{J} are charge and current densities, respectively. The vector fields $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and \mathbf{B} are assumed to have well-defined second space-time derivatives so that wave equations can be constructed.

Suppose these fields and their first and second derivatives in space-time have the Fourier representations

$$\mathbf{E}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \tilde{\mathbf{E}}(\mathbf{k}, \omega), \quad (\text{A3})$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} i\mathbf{k} \cdot \tilde{\mathbf{E}}(\mathbf{k}, \omega), \quad (\text{A4})$$

etc. The Maxwell equations in the Fourier space (\mathbf{k}, ω) then simplify to the algebraic equations

$$i\mathbf{k} \cdot \tilde{\mathbf{E}} = \tilde{\rho}/\epsilon_0, \quad i\mathbf{k} \cdot \tilde{\mathbf{B}} = 0, \quad (\text{A5})$$

$$i\mathbf{k} \times \tilde{\mathbf{E}} = i\omega \tilde{\mathbf{B}}, \quad i\mathbf{k} \times \tilde{\mathbf{B}} = \mu_0 \tilde{\mathbf{J}} - i\omega \tilde{\mathbf{E}}/c^2. \quad (\text{A6})$$

The Helmholtz theorem that decomposes any vector field $\tilde{\mathbf{E}}$ into longitudinal and transverse parts is just the BAC identity

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_{\parallel} + \tilde{\mathbf{E}}_{\perp} = \mathbf{e}_{\mathbf{k}}(\mathbf{e}_{\mathbf{k}} \cdot \tilde{\mathbf{E}}) - \mathbf{e}_{\mathbf{k}} \times (\mathbf{e}_{\mathbf{k}} \times \tilde{\mathbf{E}}), \quad (\text{A7})$$

where $\mathbf{e}_{\mathbf{k}} = \mathbf{k}/k$.

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